Alternative Technologies

Zeno's Tortoise

by David McGoveran, Alternative Technologies

A Greek philosopher by the name of Zeno of Elea (ca. 490–430 BC) is alleged to have contrived a set of paradoxes regarding, among other sense phenomena, change. Zeno's teacher Parmenides regarded change as illusory and Zeno's paradoxes were designed to support his teacher's position. Though many of Zeno's paradoxes are lost, among the most oft-related of those surviving are (1) Achilles and the Tortoise and (2) the Dichotomy. These paradoxes are particularly relevant in mathematics (sets can have infinite cardinalities), logic (well formed formulates can be of infinite length), and modern physics (spacetime is presumed continuous and so ever divisible into smaller and smaller distances), and they have been recast in those disciplines many times.

Zeno the Tortoise
Mokumegane Sculpture by Eugene Novagratsky, 2014

Achilles and the Tortoise is the story of Achilles in a footrace with Tortoise. Tortoise is given a head start of a certain distance and then each of the contestants begins to run at a constant speed, Achilles speed being much faster than that of Tortoise. After a certain time, Achilles will have run to the point at which Tortoise began and during that same time, Tortoise will have run some shorter distance. Achilles must then run for a longer time to reach that point, during which time Tortoise will have run a bit further. By the time Achilles has run this further distance, Tortoise will have run even further. The process continues so that, no matter how often Achilles reaches the point where Tortoise has already been, he can never pass Tortoise.

The Dichotomy sharpens Zeno's argument against change so that it is no longer depends on relative movement, instead using a recursive process of dividing distance into dichotomous parts. Suppose that Tortoise does not run and Achilles wishes to reach the initial head-start position of Tortoise. Achilles must reach a point that is half way between his starting position and Tortoise's position – that is, he must first run half the distance. But to reach that halfway point, he must first run a quarter of the distance. To reach that quarter distance, Achilles must first run one-eighth of the distance. This process clearly goes on forever and so Achilles can never complete the original task, there always being another task he must perform first. Furthermore, since it is impossible for Achilles to determine his first task, he cannot even begin. Hence, the Dichotomy leads us to conclude that Achilles cannot move.

In Zeno's era, the concept of space and the concept of time were treated as distinct. Although there are hints of philosophical argument about whether space was a continuum or a discretum, time seems to have been considered "instantaneous" and more of a psychological or sense phenomena than space. In modern physics, the concepts of space and time have been merged into a single conceptual structure, spacetime, in which space and time are on an equal footing as dimensions. This gives rise to the following variation on the Dichotomy, which I call Zeno's Tortoise.

Suppose that Tortoise is given the task of remaining still for a certain finite period of

time (perhaps in order to let Achilles hypothetically catch up with him). He must remain still for a time that is half way between his current time that of the finite period – that is, he must first remain still half the finite time. But to achieve that goal, he must first remain still for a quarter of the period. To remain still for that quarter of the period, he must first remain still for one-eighth of the period. This process clearly goes on forever and so Tortoise can never complete the original task, there always being another task he must perform first. Furthermore, since it is impossible for Tortoise to determine his first task, he cannot even begin to remain still. Hence, Zeno's Tortoise leads us to conclude that Tortoise cannot remain still.

It is worth noting that Zeno's paradoxes are not logical paradoxes *per se* – rather, they lead to apparently inescapable conflicts between reason and observation. Arguments that have been labeled paradoxes are of several types. While they may loosely be said to be contradictions, that is a gloss. Certainly not all contradictions are paradoxes. True paradoxes involve apparently inescapable conclusions, whether because they imply something is fundamentally wrong with, for example, (a) closely held beliefs about the nature of reality, (b) tightly held formal axioms (often suggesting inconsistent axioms that cannot be identified as such), (c) accepted proof schemata, or – in the more sophisticated paradoxes – (d) flawed approaches to logic itself. It is to this latter category that deals with self-reference, non-stratified types, failures to separate proof theory and model theory, failures to separate concepts such as object language, meta-language, and truth valuations, and so on. Please note that, though such paradoxes may imply inconsistencies, they are not simply contradictions[1].

It might be suggested that the Tortoise Paradox does not involve an infinite sequence of tasks because doing nothing (i.e., remaining motionless) is not a task.[2] Among other reasons, I reject this as being merely a *psychological* rejection since it depends on what one judges to be a "task." Certainly the term task is not well-defined in the classical versions of Zeno's paradoxes. We might define task as something requiring Achilles or the Tortoise to move, expend energy,

[1] Inconcistency is a property of a logical system and a contradiction as a relationship between statements in a logical argument. If you don't understand this difference I strongly recommend that you spend some time reading a good text on logic such as Introduction to Mathematical Logic by Stephen C. Kleene.

exert force, have a realized intent, achieve an altered condition, and so on. These properties divide into several types: those physically verifiable and those not.

So far as I am aware, all the physical properties that might be used appear to be relative either in terms of some measured value of the property or in terms of to what entity it is inferred. To make the point more clear, let's eliminate any psychological or physiological elements from the paradoxes. In particular, let us reinterpret the terms "Achilles" and "Tortoise" to be nothing more than familiar names assigned to inanimate objects. These could just as easily be fundamental particles as rocks. Now, consider movement as the defining property of a task to be completed. As is well-known from special and general relativity, motion is relative. Thus, at every instant, there is some (instantaneous inertial) frame in which either Achilles or the Tortoise is not moving.

Likewise, consider change in the separation distance between Achilles and the Tortoise. If the measured property is to be change in separation distance (a necessary measurement even if we want to infer force), then again any change is relative – we cannot know if Achilles has moved or the Tortoise has moved without appealing to psychological (and unmeasureable) properties.

With respect to force, force is reciprocal. If there exists an apparent attractive (or repulsive) force between Achilles and the Tortoise, then it may be described as equally expended by each. This has as much consequence for a task of remaining still and for moving. For example, suppose the Tortoise is to remain still and Achilles is running toward Tortoise. Let's place each inside a (possibly very large) rigid box for purposes of analysis. In order for Achilles to begin running toward Tortoise, Achilles must exert a force on the box away from Tortoise. But this means the box exerts an equal force on Tortoise. To keep the distance between Achilles and Tortoise constant, this means Tortoise must exert a counteracting force on the box. In other words, the task of "doing nothing" is also understandable as a task of "doing something" – it is

[2] I am indebted to my friend Richard Shoup for pointing out this possible objection to me.

relative.

Psychological properties such as intent are inherently subjective and cannot be verified – there is no objective test for the existence of intent or lack thereof, and no test that provides some quantitative measure of intent. I therefore suggest that a task should be understood as an objectively verifiable and specific change in some condition or property and for which there exists a mechanical procedure for determining that change. This definition eliminates what I consider to be spurious analyses of Zeno's paradoxes.

Relativity theory serves to highlight the conundrum presented by the Tortoise Paradox: It matter not whether the task in one of Zeno's arguments is to remain motionless for an interval of time or to move for some interval of space. It has been proposed that the unification of spacetime in general relativity precludes the statement of these paradoxes.[3] I disagree. All one need do is replace the classical term with the corresponding Riemannian concept (for example, replace "distance" with "spacetime interval") and the problem raised by the paradox remains: an infinitude of tasks.

With regard to mathematical methods involving limits for computation, these do not circumvent Zeno's paradoxes, but sidestep the issue. The number of mathematical tasks in going to the limit can be ignored computationally but only when there is convergence – i.e,. when the n^{th} term of the series for all integer $\mathbf{n} > \mathbf{N}$ is bounded by the limit **L** for arbitrary real number ϵ. Otherwise, even this computational procedure fails. For a physical procedure (i.e., a task), the infinite number of tasks anticipated remains problematic.

It is not my objective herein to either promote or to resolve Zeno's paradoxes. Rather, it has been my desire to demonstrate a possibly overlooked subtly of Zeno's arguments. If nothing else, Zeno's paradoxes indict forms of analysis that presume upon the top-down divisibility of concepts. The resulting infinite analytical regress is clearly a flawed procedure.

[3] Hans Reichenbach, The Direction of Time, © 1999 Dover Publications, pp. 3-18 (originally 1956).

Zeno's paradoxes continue to highlight the difficulties raised by alleged continuous or "infinite" processes. Although some have argued that modern formal methods resolve the difficulties, one must take rather troublesome ontological positions to accept them. On the other hand, although a finite discretization of spacetime seems to resolve the paradoxes, at least one modern physicist and mathematician (Hermann Weyl) denied that such approaches are consistent with empirical geometry[4]. The Tortoise Paradox, in light of Zeno's Paradoxes, leaves us in an awkward situation philosophically. Taken together, these paradoxes produce a logical contradiction, which implies the argument or proof schemata is flawed in some way. Clearly Tortoise cannot move (he is subject to the same bind as Achilles in the Dichotomy) and yet just as clearly Tortoise cannot remain still. Perhaps there exists a discrete solution?

A Grateful Acknowledgement

The Mokumegane sculture "Zeno the Tortoise" was designed and created by Eugene "Gene" Novagratsky, an artist extraordinaire, philosopher, mathematician, computer scientist, and – above all – a great friend. He presented it to the author in September of 2014. It was in conversation with Gene that the author first formulated the Tortoise Paradox.

[4] Weyl's argument depends on a construction of the Pythagorean Theorem in which the discrete is embedded in the continuum. It assumes a regularity (essentially a uniform grid of points) that is not justifiable without a continuum background. I have argued elsewhere (Foundations of a Discrete Physics, 1989) that such approaches are flawed and that the discretum must be constructed *ab initio*.